

Coupled and Extended Quintessence: theoretical differences and structure formation

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The case of a coupling between dark energy and matter (Coupled Quintessence) or gravity (Extended Quintessence) has recently attracted a deep interest and has been widely investigated both in the Einstein and in the Jordan frames (EF, JF), within scalar tensor theories. Focusing on the simplest models proposed so far, in this paper we study the relation existing between the two scenarios, isolating the Weyl scaling which allows to express them in the EF and JF. Moreover, we perform a comparative study of the behavior of linear perturbations in both scenarios, which turn out to behave in a markedly different way. In particular, while the clustering is enhanced in the considered CQ models with respect to the corresponding Quintessence ones where the coupling is absent and to the ordinary cosmologies with a Cosmological Constant and Cold Dark Matter (Λ CDM), structures in EQ models may grow slower. This is likely to have direct consequences on the inner properties of non-linear structures, like cluster concentration, as well as on the weak lensing shear on large scales. Finally, we specialize our study for interfacing linear dynamics and N-body simulations in these cosmologies, giving a recipe for the corrections to be included in N-body codes in order to take into account the modifications to the expansion rate, growth of structures, and strength of gravity.

I. INTRODUCTION

The increasing amount of cosmological observations has been recently pointing out that the Universe is nearly spatially flat, with an expansion rate of about 70 km/s/Mpc and with a 76% contribution to the total energy density due to a non-clustered, negative pressure component responsible for the acceleration in the cosmic expansion, known as dark energy. The energy level of this component is markedly lower than almost all known physical quantities, except perhaps the neutrino mass differences, and this motivated big efforts in order to understand its physical properties. Interesting hints have been obtained studying scenarios in which the dark energy is dynamic, i.e. described by an evolving and fluctuating scalar field, the Quintessence. In particular, several investigations have been devoted to study the possibility that the dark energy might be coupled to other entities in the Universe, altering the dynamics of the Quintessence field itself as well as the evolution of structure formation. It is then essential to understand the impact of such a coupling on both cosmological expansion and perturbation dynamics, in view of a clear prediction of observable effects for the next to come experiments.

Two main efforts have been recently carried out in a nearly parallel way. On one side, within the usual framework of General Relativity (Einstein Frame, EF) the effect of a coupling between dark energy and dark matter (Coupled Quintessence, CQ) has been deeply investigated both in the linear [1] and non-linear regimes [2, 3], including the effects of such models on the Cosmic Microwave Background (CMB) [4, 5, 6], on Supernovae [7, 8, 9], on the matter power spectrum [10], on structure formation [11, 12], also via N-body simulation [13], as well as different choices of the coupling [14, 15, 16, 17, 18, 19]. The possibility of an interaction between dark energy and neutrinos has also been investigated [20, 21], leading recently to the proposal of growing matter scenarios [22, 23, 24]. Instabilities within coupled dark energy theories have also been addressed [25, 26, 27].

On the other side, a wide variety of analysis has been carried out for evaluating the possibility that the gravity perceived by dark energy and matter deviates from General Relativity (GR). In this latter case, extensions of GR have been considered, in which the dark energy might derive from a non-minimal interaction to gravity via an explicit coupling between Quintessence and the Ricci scalar (Jordan Frame, JF). This is the case of scalar tensor theories [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40], which are known as Extended Quintessence (EQ) scenarios in the framework of dark energy. One of the most interesting results in this framework consists in the so called ‘gravitational dragging’ [36]: the coupling to gravity introduces an effective potential in which the quintessence scalar field rolls, acquiring a dynamics which is a purely gravitational effect. Eventually, EQ energy density perturbations can be dragged to cluster through the coupling with the gravitational potential activated by the dark matter, its density perturbations reaching non-linearity at scales and redshifts relevant for the structure formation process. In particular, the growth of dark energy perturbations in the non-linear regime within EQ has been addressed in [40].

Furthermore, in the case in which, in the EF, dark energy couples universally to all matter fields, the two parallel investigations can also be conformally related by Weyl scaling, which allows to go univoquely from one representation to the other. Although the resulting physical effects must be equivalently present in both frames, some features might more naturally appear in one representation than in the other and viceversa.

In this paper, we identify the theoretical relation between the simplest and most popular examples of both coupling choices for CQ and EQ. On the basis of this analysis, we work out the main phenomenological differences in terms of the growth of linear perturbations, which turn out to behave in a markedly different way for the two models considered. Finally, we propagate our investigation to the inputs required by N-body simulations in view of future applications, giving a table of the modifications required in both scenarios for taking into account the modified growth rate of perturbations and the strength of gravity. This work is organized as follows: we first devote sec.(II) to Weyl scaling, recalling how it is possible to relate EQ to CQ via a conformal transformation; afterwards, in (sec.III) we investigate the background dynamics, comparing CQ in the simple case of a constant coupling (III A) and EQ in scalar tensor theories (III B); we then proceed by studying linear perturbations (sec.IV), again presented within both cosmological approaches (IV A, IV B). We will then point out in particular, how in the newtonian limit (V) the Euler equation is modified (sec.V A, sec.V B), in view of N-body simulations to be realized in both frameworks and for which we specify the required corrections (sec.V C). Finally in (sec.VI) we compare the results and set our conclusions.

II. WEYL SCALING

Before handling a parallel analysis of the background and linear perturbations in CQ and EQ models, it is compelling to clarify the fact that the two frameworks are strictly related through a conformal transformation called Weyl scaling [41, 42, 43, 44, 45, 46]. It is important to stress that altering General Relativity via scalar tensor theory (Jordan frame, JF) is equivalent to coupling a scalar field universally with all matter fields within General Relativity (Einstein frame, EF). Although the two reference frames must lead to identical observable effects, the description of the same model can in fact rely on different and unequally simple equations when seen in the two frames. In this sense, simplicity and naturalness of one frame are not in general preserved in the other one. Going from one frame to the other and viceversa according to the model, profiting of this frame equivalence, can therefore represent a valuable tool to easily explore a wider set of choices for the coupling functions and the potentials involved: a nasty choice in one frame might, in other words, look much simpler in the other frame, allowing us to easily stress remarkably new features, more difficult to achieve in the frame of origin. A frame-independent approach has also been considered [45] leading to a reformulation of the Boltzmann equation and linear perturbation theory in terms of frame-independent quantities. Authors usually refer to CQ models when they consider a quintessence scalar field coupled, in the Einstein frame, to the dark matter fields. The action considered in this case usually appears as follows:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} R - \frac{1}{2} Z(\phi) \phi^{;\mu} \phi_{;\mu} - U(\phi) - m(\phi) \bar{\psi} \psi + \mathcal{L}_{\text{kin},\psi} \right], \quad (1)$$

where g is the determinant of the background metric, R is the Ricci scalar, $\kappa = 8\pi G$ where G represents the gravitational constant, as found in Cavendish like experiments. In the assumption of flat Friedmann Robertson Walker (FRW) cosmologies, the line element can be written as $ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j$, where $a(t)$ is the scale factor and t represents the cosmic time variable. The choice of $m(\phi)$ specifies the coupling to ψ matter fields while $\mathcal{L}_{\text{kin},\psi}$ includes kinetic contributions from all components different from ϕ .

In the Jordan frame, a scalar tensor theory in which EQ holds is in general described by the following action:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} f(\phi, R) - \frac{1}{2} Z(\phi) \phi^{;\mu} \phi_{;\mu} - U(\phi) - m_0 \bar{\psi} \psi + \mathcal{L}_{\text{kin},\psi} \right]. \quad (2)$$

The scalar field ϕ characterizing the generalized gravitational interaction has its kinetic energy and potential specified by $Z(\phi)$ and $U(\phi)$, respectively. In this case the mass m_0 is a constant and does not depend on the scalar field ϕ . Note that we consider here natural units, $c = 1$. Compared to general relativity, the Lagrangian has been generalized by introducing an explicit coupling between the Ricci scalar and the scalar field, achieved by replacing the usual Ricci scalar R with the function $f(\phi, R)$. This new term, which has the effect of introducing a spacetime dependent gravitational constant, may either be interpreted as an explicit coupling between the quintessence field ϕ and gravity (or equivalently, in the Einstein frame, between dark energy and matter), or as a pure geometrical modification of general relativity admitting a non-linear dependence on R [59]. For simplicity, the classes of theories in which $f(\phi, R)$ assumes the simple form $f(\phi, R)/2 = \kappa F(\phi) R/2$ have been widely considered in the context of dark energy cosmologies.

Weyl scaling consists in a conformal transformation of the metric which, joined to a redefinition of matter fields, allows to rewrite action (2) into (1) or viceversa:

$$g_{\mu\nu} = A^2(\phi) \tilde{g}_{\mu\nu} \quad (3)$$

$$g^{\mu\nu} = A^{-2}(\phi)\tilde{g}^{\mu\nu} , \quad (4)$$

$$\sqrt{-g} = A^4(\phi)\sqrt{-\tilde{g}} , \quad (5)$$

$$R = [A(\phi)]^{-2}\{\tilde{R} - 6\tilde{g}^{\mu\nu}(\ln A)_{;\nu}(\ln A)_{;\mu}\} , \quad (6)$$

$$\psi = A^{-3/2}(\phi)\tilde{\psi} , \quad (7)$$

where we have used the $\tilde{}$ to identify quantities in the EF and distinguish them from those in the JF. Note also that the scaling factor $A(\phi)$ is related to the coupling $F(\phi)$ via the following relation:

$$A^2(\phi) = \frac{1}{\kappa F(\phi)} . \quad (8)$$

When applying Weyl scaling to the action (2) with $Z(\phi)=1$ for simplicity, we obtain that the rescaled action appears to be

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2\kappa} \tilde{R} - \frac{1}{2} \tilde{Z}(\phi) \tilde{\phi}^{;\mu} \tilde{\phi}_{;\mu} - \tilde{U}(\phi) - m(\phi) \tilde{\psi} \tilde{\psi} + \tilde{\mathcal{L}}_{\text{kin},\tilde{\psi}} \right] , \quad (9)$$

where

$$\tilde{Z}(\phi) = A^2(\phi) \left(1 + \frac{6}{\kappa} \frac{A_{,\phi}^2}{A^4(\phi)} \right) , \quad (10)$$

$$\tilde{U}(\phi) = A^4(\phi) U(\phi) , \quad (11)$$

and both the mass and the kinetic term for matter fields now also depend on ϕ :

$$m(\phi) = m_0 A(\phi) \quad (12)$$

$$\tilde{\mathcal{L}}_{\text{kin},\tilde{\psi}}(\tilde{\psi}, \phi, \tilde{g}^{\mu\nu}) = i \tilde{\psi} \tilde{\gamma}^\mu \nabla_\mu \tilde{\psi} + i \frac{3}{2} \frac{A_{,\phi}}{A(\phi)} \tilde{\psi} \tilde{\gamma}^\mu \nabla_\mu \tilde{\psi} . \quad (13)$$

Note that the non standard kinetic term \tilde{Z} in (10) is always positive and one could further re-absorb it by redefining the scalar field ϕ implicitly into φ in such a way that:

$$\left(\frac{\partial \varphi}{\partial \phi} \right)^2 \equiv \tilde{Z}(\phi) . \quad (14)$$

This allows to rewrite action (9) in the usual form:

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2\kappa} \tilde{R} - \frac{1}{2} \varphi^{;\mu} \varphi_{;\mu} - \tilde{U}(\varphi) - m(\varphi) \tilde{\psi} \tilde{\psi} + \tilde{\mathcal{L}}_{\text{kin},\tilde{\psi}} \right] . \quad (15)$$

We have now all the means to find out, given a certain action (2) in the Jordan frame, which is the equivalent action (15) in the Einstein frame. In the parallel analysis that will follow in this paper, we will eventually refer to two particular choices of the coupling in CQ and EQ scenarios. In particular, we will consider two of the simplest and most popular choices, that is to say a quadratic coupling in the Jordan frame for EQ [34, 40, 43, 47] and an exponential coupling in the Einstein frame for CQ [1, 2]. In order to have a feeling of a possible comparison between the two different models, we illustrate here how they look like in the same, Einstein, frame, after Weyl scaling a quadratic EQ model.

In the case of EQ with a quadratic coupling, we will work with so called ‘non-minimally coupled’ theories, widely used in previous works on this topic (see [34] and references therein), in which $F(\phi)$ is the sum of a dominant constant term plus a piece depending on ϕ :

$$F(\phi) \equiv \frac{1}{8\pi G^*} + \xi \phi^2 . \quad (16)$$

Here $\kappa_* \equiv 8\pi G_*$, where G_* represents the ‘bare’ gravitational constant [42], which is in general different from G and is set in such a way that locally $1/\kappa + \xi\phi_{local}^2 = 1/(8\pi G)$ in order to match local constraints on General Relativity determined by a combination of G_* and the local value of $f(\phi, R)$ such that $G \equiv G_* \frac{R}{f}|_{local}$. In order to recover the equivalent action in the Einstein frame, we apply equations (8, 10, 11, 12), thus obtaining action (9) in which:

$$\tilde{Z}(\phi) = \frac{1}{\frac{G}{G_*} + \kappa\xi\phi^2} + 6\kappa\xi^2\phi^2, \quad (17)$$

$$\tilde{U}(\phi) = \frac{1}{\left[\frac{G}{G_*} + \kappa\xi\phi^2\right]^2} U(\phi), \quad (18)$$

$$m(\phi) = \frac{m_0}{\sqrt{\left|\frac{G}{G_*} + \kappa\xi\phi^2\right|}}. \quad (19)$$

Notably, in this frame the coupling to matter fields, the kinetic term and the potential do not look as straightforward as in the original action.

Another very popular choice is represented by Induced Gravity (IG) [34, 43, 47, 48], in which only the quadratic coupling

$$F(\phi) = \xi\phi^2 \quad (20)$$

is considered and no constant term is present. It can be easily seen, by applying equations (8, 10, 11, 12) that in this case we recover action (9) in which:

$$\tilde{Z}(\phi) = \frac{1}{\kappa\xi\phi^2} (1 + 6\xi), \quad (21)$$

$$\tilde{U}(\phi) = \frac{1}{\kappa^2\xi^2\phi^4} U(\phi), \quad (22)$$

$$m(\phi) = \frac{m_0}{\sqrt{\kappa\xi\phi^2}}. \quad (23)$$

In this case it is also straightforward to solve equation (14) analytically, obtaining

$$\phi = De^{C\varphi}, \quad (24)$$

for $\phi > 0$, where C and D are constants and we have defined $C \equiv \sqrt{\xi/(1+6\xi)}$. The latter expression, substituted in (22, 23) allows to recover action (15) in which:

$$\tilde{U}(\varphi) = \frac{D^4}{\kappa^2\xi^2} e^{-C\varphi} U(\phi(\varphi)), \quad (25)$$

$$m(\varphi) = \frac{m_0 D}{\sqrt{\kappa\xi}} e^{-C\varphi}. \quad (26)$$

Applying Weyl scaling to IG theories, leads, as expected [43, 48], to CQ models in which the mass of matter fields depends exponentially on the scalar field φ . We therefore recover the simplest and most popular choice in CQ investigations [1] and we will ourselves adopt this choice in this paper when dealing with CQ in the Einstein frame. Furthermore, since $U(\phi)$ is usually almost constant in order to match the present constraints on the amount of Quintessence, the transformed potential in the Einstein frame behaves as an exponential.

As appears clear from the previous analysis, IG theories in the Jordan frame (or equivalently CQ models in the Einstein frame with an exponential coupling in the Lagrangian) have a crucial difference from the non-minimal coupling choice described by eq.(16) in the Jordan frame. Indeed, IG (or CQ with an exponential coupling) forces the coupling constant ξ to be positive in order to get the right sign for attractive gravity in action (2); on the contrary, the choice (16) also allows negative values of ξ and can therefore lead to an opposite correction to background and perturbation dynamics, as we will see in detail in the following section.

III. BACKGROUND DYNAMICS

In this Section we define the CQ and EQ models we study in the following, and review their dynamics for what concerns the cosmic expansion, highlighting the aspects which will be relevant for the behavior of linear perturbations.

A. Coupled dark energy

We will follow as much as possible previous works on this topic [1, 3], considering the simplest models in which the dark energy scalar field and CDM are coupled via a constant coupling appearing in the conservation equations. The equations used to describe the background evolution of each component (α) involved in the interaction follow from the consideration [49] that the coupling can be treated as an external source acting on each stress energy tensor $T^\mu_{\nu(\alpha)}$ in such a way that the total stress energy tensor is conserved:

$$T^\mu_{\nu;\mu(c)} = -C_c T_c \phi_{;\nu} \quad (27)$$

$$T^\mu_{\nu;\mu(\phi)} = C_c T_c \phi_{;\nu} , \quad (28)$$

where C_c is a constant, T_c is the trace of $T^\mu_{\nu(c)}$ and the subscripts (c) , (ϕ) stand for CDM and dark energy respectively. Baryons do not couple with dark matter or dark energy. The constant coupling term here used can be achieved starting from the following Lagrangian:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} R - \frac{1}{2} \phi^{;\mu} \phi_{;\mu} - U(\phi) - m(\phi) \bar{\psi} \psi + \mathcal{L}_{\text{kin},\psi} \right] , \quad (29)$$

in which the mass of cold dark matter fields depends exponentially on ϕ [58]:

$$m(\phi) = m_0 e^{-C_c \phi} , \quad (30)$$

thus corresponding, in the JF, to IG cosmologies, as seen at the end of the previous section. Here, for simplicity, we do not include the baryonic component.

Equations (27,28) specialized for $\nu = 0$, provide the conservation equations for the energy densities of each component, given by $\rho'_c = -3\mathcal{H}\rho_c - C_c \rho_c \phi'$ and $\rho'_\phi = -3\mathcal{H}h_\phi + C_c \rho_c \phi'$ for the two coupled species and $\rho'_r = -4\mathcal{H}\rho_r$ since $C_r = 0$ for radiation. ρ_r takes into account the radiation contribution, including both photons and neutrinos, both decoupled from the other species. We have further defined $h_\phi \equiv p_a + \rho_a = \phi'^2/a^2$ where the latter equality comes from the expression of the energy density ρ_ϕ defined from the $(0,0)$ component of the stress energy tensor present in (28) and therefore given by the usual

$$\rho_\phi = \frac{1}{2} \frac{\phi'^2}{a^2} + U(\phi) , \quad (31)$$

where $U(\phi)$ is the potential in which the dark energy scalar field rolls. Note that ρ_ϕ is still equal to the standard one characterizing uncoupled quintessence models since, as we have seen, the coupling is provided in (27, 28) as an external contribution to the standard stress energy tensor; nevertheless, unlike what happens in the uncoupled quintessence case, only the sum $\rho_c + \rho_\phi$ is conserved while ρ_ϕ is not conserved by itself.

Analogously, the pressure of the quintessence scalar field is defined as:

$$p_\phi = \frac{1}{2} \frac{\phi'^2}{a^2} - U(\phi) . \quad (32)$$

For what concerns the background dynamics, the evolution of the scalar field is described by the Klein Gordon equation, which reads as follows:

$$\phi'' + 2\mathcal{H}\phi' + a^2 U_{,\phi} = a^2 C_c \rho_c , \quad (33)$$

where $U_{,\phi}$ is the derivative of the potential with respect to ϕ .

The CDM conservation equation can be formally integrated out

$$\rho_c = \frac{\rho_{c0}}{a^3} e^{-C_c(\phi-\phi_0)} \quad (34)$$

giving the dependence of ρ_c from the scale parameter, once that the dynamics of $\phi(a)$ is known. The evolution in time of the scale factor is provided by the usual Friedmann equation, so that:

$$\mathcal{H}^2 \equiv a^2 \frac{\rho_c + \rho_\phi + \rho_r}{3} . \quad (35)$$

For convenience we have defined $\hat{\rho} \equiv 8\pi G\rho = \rho/M^2$ and $\hat{\phi} \equiv \phi/M$ in units of the reduced Planck mass $M^2 \equiv 1/8\pi G = M_P^2/8\pi$ where G is the gravitational constant. The coupling constant $\hat{C} \equiv CM$ is therefore dimensionless; we have then omitted the $\hat{}$ and redefined $\hat{\rho} \equiv \rho$, $\hat{\phi} \equiv \phi$ and $\hat{C} \equiv C$ for simplicity.

A detailed analysis of the background has already been addressed in [1] in the case in which $U(\phi)$ is an exponential and in [3, 8, 13] for a more general treatment. In particular, the critical points of the trajectories in the phase space of the system have been widely investigated in [1]. We just recall here that in the case in which both the potential $U(\phi)$ and radiation are neglected, it is straightforward to find the critical points of the system, corresponding to $d\phi_{crit}/d\alpha = (-\sqrt{6}, +\sqrt{6}, 2C_c)$ where we have defined $\alpha = \log[a(\tau)]$. The third critical point is the only one providing a stable attractor during matter dominated era (MDE) [1]¹ along which the evolution of the dark energy scalar field follows the trajectory

$$\phi(a) = 2C_c \log a + \phi_0, \quad (36)$$

where we have put today's value of the scale parameter equal to $a_0 = 1$. Substituting in (34), we get:

$$\rho_c = \frac{\rho_{c0}}{a^3} a^{-2C_c^2}. \quad (37)$$

We can now make use of the Friedmann equation, specified in the case in which there is no radiation and the potential is negligible, to obtain the scale parameter dependence on the conformal time during MDE:

$$a_{MDE}(\tau) \propto \tau^{\left[\frac{2}{1+2C_c^2}\right]} \quad (38)$$

as well as $\mathcal{H}(\tau)$:

$$\mathcal{H}_{MDE}(\tau) = \frac{2}{1+2C_c^2} \frac{1}{\tau}, \quad (39)$$

or equivalently, in terms of the cosmic time $a_{MDE}(t) \propto t^{2/(3+2C_c^2)}$ and $H_{MDE}(t) = 2/[(3+2C_c^2)t]$. We stress here that, as we can see from the four latter expressions, the effect of the coupling on the Hubble parameter and on the scale factor during MDE, when the potential is negligible, does not depend on the sign of the constant C_c , both of them being only a function of C_c^2 . With this choice of the coupling, we can thus only have a one direction effect, being that of slowing down the universe expansion with respect to the standard case in which $C_c = 0$.

We can numerically solve the equations describing the background, in order to obtain the full evolution of the energy densities of both cold dark matter and the scalar field, as shown in fig.(1). In the left panel, the energy densities of a cosmological constant (dotted), radiation (long dashed), CDM (solid) and coupled quintessence (dashed) in the case in which $C = 0.05$ have been plotted. We have used an inverse power law shape for the potential $U(\phi)$ which was first proposed in [50] and often used in quintessence models since then [51, 52]:

$$U(\phi) = U_0 \left(\frac{\phi_0}{\phi} \right)^\lambda, \quad (40)$$

where ϕ_0 is the value of the scalar field today and λ will be typically fixed to ~ 0.2 in order to have a flat potential and a consequent equation of state for dark energy which is close to -1 today, in order to be consistent with observations [53, 54, 55]. Another popular choice is given by an exponential potential [1, 57, 58], though we are not interested here in the explicit expression of $U(\phi)$, which does not modify our considerations. The amplitude of the potential (40) and the initial amount of CDM are fixed in order to have their values today again in agreement with the present constraints [53, 54, 55]. As shown in fig.(1) (left panel), the presence of the coupling has a drastic effect on the behavior of the scalar field energy density. In the uncoupled case quintessence behaves pretty much as a cosmological constant due to the rather flat potential. If however the coupling to CDM is active, ϕ accelerates and the dark energy density is attracted by the evolution of the species to which it is coupled: in particular, after a period of transition due to the particular choices of the initial conditions, $\bar{\rho}_\phi$ is dragged by CDM to follow a trajectory which strictly mimics the CDM path, therefore behaving as an effective pressureless component. We will refer to this effect as ‘dark matter dragging’ in analogy to the ‘gravitational dragging’ first discussed in [36] in which the coupling of the scalar field to gravity via a modification of General Relativity imprinted a similar pattern to the scalar field dynamics, both

¹ The three critical points correspond to the last three critical points in table 1 in [1].

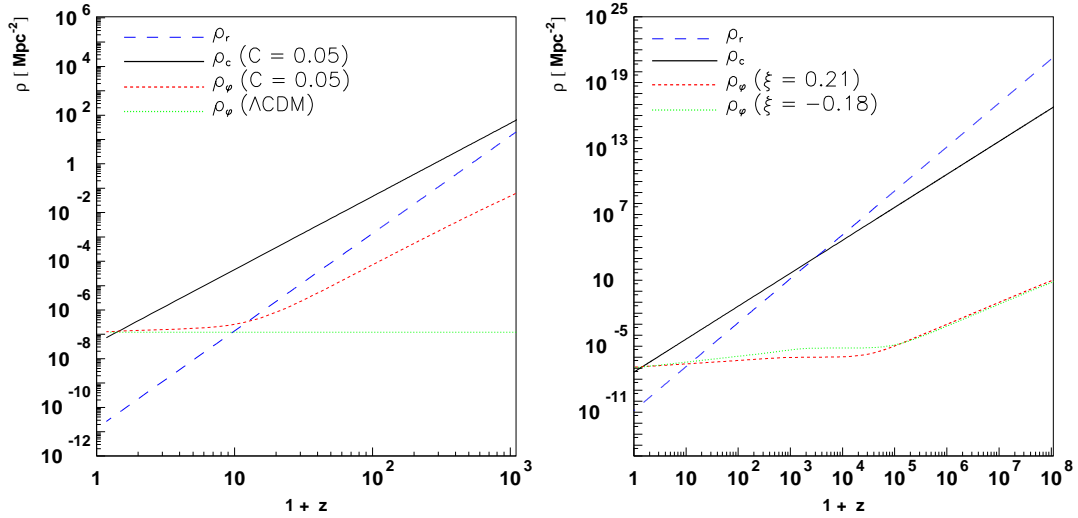


FIG. 1: Energy densities of CDM and dark energy. The left panel shows the case of coupled dark energy for $C = 0.05$ as well as for $C = 0$ vs $1 + z$, where z is the redshift. The right panel shows the case of extended quintessence for both positive and negative coupling corresponding to $\omega_{JBD0} \sim 30$. During MDE the energy density of the scalar field (the non conserved one) is more enhanced in the case of coupled dark energy than in extended quintessence, in which the dynamics of the R-boost has its major influence during RDE. The plotted densities are in units of Mpc^{-2} .

on background and on perturbations. In this scenario, the matter scaling is also modified by the coupling with the field; the resulting regime of cosmological expansion driven by the field and CDM tracking each other, is known as ϕCDM era, see [3] and references therein.

In both models here considered, ϕ rolls down an effective potential provided by the coupling to dark matter (right hand side of eq.33) or by modified gravity (right hand side of equation 44, next paragraph). Note however that despite the analogy with gravitational dragging, during dark matter dragging the dark energy scalar field does not exactly slow roll on the effective potential, as it happens in the gravitational dragging case: when the coupling to dark matter is considered, ϕ acquires a strong acceleration from the dark matter field, that makes terms like ϕ'' give a non negligible contribution in equation (33), an occurrence which is absent in the EQ case, as we now discuss.

B. Extended quintessence

As illustrated in the first section, scalar tensor theories are described by action (2) that we rewrite here for convenience:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa} f(\phi, R) - \frac{1}{2} \phi^{\mu} \phi_{;\mu} - U(\phi) - m_0 \bar{\psi} \psi + \mathcal{L}_{\text{kin}, \psi} \right]. \quad (41)$$

We have chosen $Z(\phi) = 1$ to simplify the notation. We further set $f(\phi, R) = \kappa F(\phi)R$ so that the function $F(\phi)$ fully specifies the explicit coupling to gravity via its dependence on some scalar field ϕ . When the additional scalar field introduced in the coupling plays today the role of the dark energy field, leading to an accelerated expansion of the universe, we refer to it as extended quintessence, see [37] and references therein. The limit of general relativity is achieved with $\omega_{JBD} \gg 1$ where ω_{JBD} is defined as

$$\omega_{JBD} \equiv \frac{F}{F_\phi^2}, \quad (42)$$

and F_ϕ is the derivative of F with respect to ϕ . The current constraints set the present value ω_{JBD0} to be larger than 40000 in the Solar System [60], and greater than about 120 from cosmology [61]; being on markedly different spacetime scales, the two are considered as independent and complementary [62]. Within these theories, the Friedmann equation describing the expansion of the Universe is modified with respect to the usual case:

$$\mathcal{H}^2 = \left(\frac{a'}{a} \right)^2 = \frac{a^2}{3F} \left(\rho_{\text{fluid}} + \frac{1}{2} \frac{\phi'^2}{a^2} + U(\phi) - \frac{3\mathcal{H}F'}{a^2} \right). \quad (43)$$

Besides the extra term at the end of (43), which is usually quantitatively negligible, the most important effect to point out is the presence of the multiplying factor $1/F$ substituting the usual $8\pi G$; as a consequence, extended quintessence behave as theories in which gravity depends on a varying function and not on a gravitational constant anymore. Note that now all fluids but ϕ satisfy the usual conservation equation $\rho'_{\alpha} = -3\mathcal{H}h_{\alpha}$ including CDM, which now scales as $\rho_c = \rho_{c0}/a^3$. The evolution of the scalar field is explicited by the Klein Gordon equation, which again gets contribution from the coupling to gravity, enhancing the background dynamics of ϕ and giving rise to the R-boost effect illustrated in [63]

$$\phi'' + 2\mathcal{H}\phi' + a^2 U_{,\phi} = \frac{a^2}{2} F_{,\phi} R, \quad (44)$$

where $F_{,\phi}$ and $U_{,\phi}$ are the derivatives of the coupling $F(\phi)$ and of the quintessence potential $U(\phi)$ with respect to ϕ . The term on the right hand side has the effect of altering the potential $U(\phi)$ into an effective potential in which the scalar field rolls. This effect, responsible for the gravitational dragging [36], is analogue to the one produced by the right hand side of equation (33) due to the coupling to matter fields, within the Einstein frame, as shown in fig.(1 right panel): here we plot the energy densities of radiation (long dashed), CDM (solid) and EQ in the case in which the coupling to gravity is positive (dashed) or negative (dotted). The energy density of the scalar field has a similar behavior during RDE (R-boost effect), independently of the sign of ξ , while the two patterns detach a bit during MDE. The plot has been obtained solving numerically the equations for the background in the case of non-minimal coupling, with $F(\phi)$ given by (16). Again, the effect of the coupling on the background evolution is that of enhancing the amount of dark energy in the past, due to the gravitational dragging set on by the additional term in the Klein Gordon equation (44). Though in the quadratic case the gravitational dragging shows in particular during RDE, note that the time at which the dragging ends depends on the choice of the model and on the values of the coupling parameter. An exponential coupling in the lagrangian would have a bigger effect, lasting up to MDE and more recent times [39]. Although the phenomenology and the energy density scaling is clearly analogous for the CQ and EQ models plotted in fig.(1), it is important to stress here that the sign of the coupling leads however to a different correction in the Hubble expansion parameter. Indeed, while in coupled dark energy the Hubble parameter does not depend on the sign of the coupling constant during MDE (eq.39), in extended quintessence a positive and a negative coupling lead to different sign corrections, either enhancing or reducing the Hubble parameter with respect to the standard Λ CDM case. The effect is shown in fig.(2): here we plot the hubble parameter versus redshift for Λ CDM (solid), coupled quintessence with $C = 0.1$ (dotted) and extended quintessence with a positive (long dashed) or negative (dashed) coupling. For the chosen coupled quintessence model, with a constant coupling, the hubble parameter is bigger in the past than the usual Λ CDM case, independently of the sign of the coupling constant. The chosen extended quintessence case, instead, can lead to both higher and lower values in the past, depending on the sign of the coupling. Note also that the switch in sign does not lead to perfectly opposite contributions: when the coupling is negative the effect is bigger than in the case of a positive coupling with the same absolute value. A reason for this to happen might be that in the first case the extra term in the Klein Gordon equation adds to the usual potential, favoring an easier enhancement of the dynamics of the field; on the contrary, for a positive constant, the extra term contrasts the effect of $V(\phi)$, making it more difficult to enhance the dynamics of the field.

Finally, we recall that in this context, the Ricci scalar can be written in terms of the cosmological content of the Universe

$$R = -\frac{1}{F} \left[-\rho_{fluid} + 3p_{fluid} + \frac{\phi'^2}{a^2} - 4U + 3 \left(\frac{F''}{a^2} + \frac{2\mathcal{H}F'}{a^2} \right) \right], \quad (45)$$

where ρ_{fluid} and p_{fluid} are the energy density and pressure summed up over all possible cosmological components but ϕ .

IV. LINEAR PERTURBATIONS

Here we perform a comparative analysis of the behavior of linear perturbations in the two scenarios. As we show in a moment, the main difference highlighted in the previous section persist at this level, as in the Newtonian limit which is outlined below.

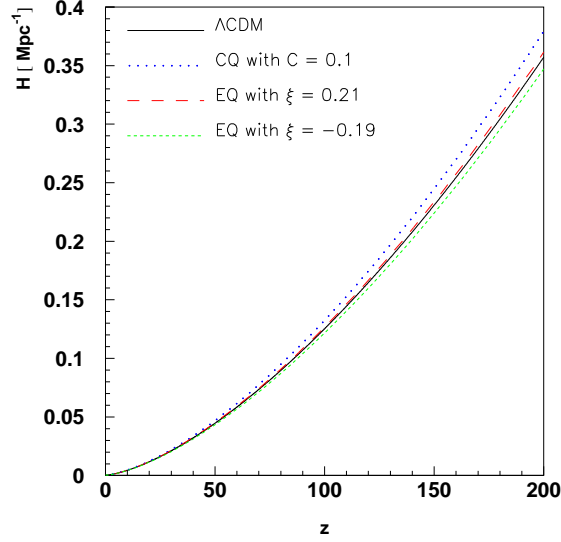


FIG. 2: Hubble parameter vs redshift for CQ and EQ with a non minimal coupling (for positive and negative couplings). The case of Λ CDM is also shown for reference. The value of ξ corresponds to $\omega_{JBD0} \sim 30$. H is in units of Mpc^{-1} .

A. Coupled dark energy

The linearized perturbation equations can be derived, following the notation of [49], from

$$\delta(T^\mu_{\nu;\mu}{}_{(\alpha)}) = \delta(Q_\nu{}_{(\alpha)}) , \quad (46)$$

with $Q_\nu{}_{(\alpha)} = (-aQ_\alpha, \mathbf{0})$ being the source term for the α -component. In particular, comparing with eq.(27) and eq.(28), for the case of a constant coupling here considered the source is defined as:

$$Q_\phi \equiv \frac{C_c \bar{\rho}_c \bar{\phi}'}{a} \quad (47)$$

$$Q_c \equiv -Q_\phi . \quad (48)$$

For $\nu = 0$ one gets the perturbed energy conservation equations, while for $\nu = i$ (spatial index) we can recover the first order Euler equations. In the Newtonian gauge in which the non-diagonal metric perturbations are fixed to zero, the linearized conservation equations for the matter component and for the scalar field, respectively, read as follows:

$$\delta\rho'_c + 3\mathcal{H}\delta\rho_c + \bar{\rho}_c k v_c + 3\bar{\rho}_c \Phi' = -C_c(\bar{\rho}_c \delta\phi' + \bar{\phi}' \delta\rho_c) , \quad (49)$$

$$v'_c + (\mathcal{H} - C_c \bar{\phi}') v_c = -k(\Phi + C_c \delta\phi) , \quad (50)$$

$$\delta\rho'_\phi + 3\mathcal{H}(\delta\rho_\phi + \delta p_\phi) + k\bar{h}_\phi v_\phi + 3\bar{h}_\phi \Phi' = C_c(\bar{\rho}_c \delta\phi' + \bar{\phi}' \delta\rho_c) , \quad (51)$$

$$\bar{h}_\phi v'_\phi + (\bar{h}'_\phi + 4\mathcal{H}\bar{h}_\phi) v_\phi = k\delta p_\phi - k\bar{h}_\phi \Phi + C_c k \bar{\rho}_c \delta\phi , \quad (52)$$

$$\delta\rho'_r + 4\mathcal{H}\delta\rho_r + k\frac{4}{3}\bar{\rho}_r v_r + 4\bar{\rho}_r \Phi' = 0 , \quad (53)$$

$$\bar{\rho}_r v'_r - \frac{k}{4}\delta\rho_r = -k\bar{\rho}_r \Phi , \quad (54)$$

where k is the wavenumber, v_α is the velocity perturbation (along the direction of \vec{k}) for the α species and Φ is the gravitational potential coming from the metric perturbation; in particular $\Phi = -\Psi$ if we neglect the anisotropic

stress. Note that the latter is rigorously zero for the present scenario, as opposed to the EQ case as we will see in a moment. Thus the only source of that here is represented by neutrinos, due to their viscosity for having decoupled in a markedly relativistic regime. Also note that the two latter equations hold only if we neglect baryons, otherwise Thomson scattering with photons should be taken into account. A more convenient way to express equation (49) is to rewrite it in terms of $\delta_c \equiv \delta\rho_c/\rho_c$ as

$$\delta'_c + kv_c + 3\Phi' = -C_c\delta\phi'. \quad (55)$$

The gravitational potential Φ is related to the background and perturbed densities and velocities of the various components via the following expression:

$$\Phi = \frac{a^2}{2k^2}\rho\Delta = \frac{a^2}{2k^2} \left[\delta\rho_c + \delta\rho_\phi + \delta\rho_r + 3\frac{\mathcal{H}}{k} \left(\bar{\rho}_c v_c + \frac{4}{3}\bar{\rho}_r v_r + \bar{h}_\phi v_\phi \right) \right]. \quad (56)$$

Or, equivalently, in a more compact form and in terms of the $\Omega_{(\alpha)}$ as

$$\Phi = \frac{3}{2}\lambda^2 \left[\sum_i \Omega_i (\delta_i + 3(1 + \omega_i)\lambda v_i) \right], \quad (57)$$

where we have defined $\lambda \equiv \mathcal{H}/k$ and where $\Omega_c = (a^2\rho_c)/(3\mathcal{H}^2)$ and the sum is extended to all components, including dark energy. Note that dark energy perturbations contribute to the gravitational potential and therefore on the dark matter perturbations through eq.(55).

Deriving with respect to the conformal time, making use of equations (49, 50, 51, 52) as well as of the background conservation equations and substituting the expression of Φ given by (56) we get the following simplified expression for Φ' :

$$\Phi' = -\mathcal{H}\Phi - \frac{a^2}{2k}(\bar{\rho}_c v_c + \frac{4}{3}\bar{\rho}_r v_r + \bar{h}_\phi v_\phi). \quad (58)$$

The energy density perturbation for the dark energy scalar field can be expressed in terms of $\delta\phi$, ϕ and their derivatives in the following way:

$$\delta\rho_\phi \equiv \rho_\phi\delta\phi = \frac{1}{a^2} [\phi'(\delta\phi)' + a^2 U_{,\phi}\delta\phi + (\phi')^2\Phi]. \quad (59)$$

We recall that the latter expression, holding in the newtonian gauge, can be derived from $\delta T^0_{0(\phi)}$ of the dark energy scalar field when comparing it to the usual one for a fluid of pressure p_ϕ and energy density ρ_ϕ (or, equivalently, directly perturbing (31)). Analogously, for the velocity perturbation in the newtonian gauge, one has

$$h_\phi v_\phi = \frac{k}{a^2}\phi'\delta\phi, \quad (60)$$

which can be derived from the $\delta T^0_{j(\phi)}$ component of the perturbed stress energy tensor. Making use of the two latter expressions, Φ and Φ' can be rewritten as

$$\Phi = \frac{[\phi'(\delta\phi)' + 3\mathcal{H}\phi'\delta\phi + a^2 U_{,\phi}\delta\phi + 3\mathcal{H}^2 [\Omega_c(\delta_c + 3\lambda v_c) + \Omega_r(\delta_r + 4\lambda v_r)]]}{2k^2 - (\phi')^2}. \quad (61)$$

$$\Phi' = -\frac{1}{2} \left[\phi'\delta\phi + 2\mathcal{H}\Phi + 3\frac{\mathcal{H}^2}{k} \left(\Omega_c v_c + \frac{4}{3}\Omega_r v_r \right) \right]. \quad (62)$$

Note also that due to the definition of the energy density and pressure of the quintessence scalar field (31, 32) the perturbed pressure appearing in eq.(52) is simply equal to:

$$\delta p_\phi = \delta\rho_\phi - 2U_\phi(\phi)\delta\phi. \quad (63)$$

As regard to the dark energy scalar field perturbation, $\delta\phi$ can be either written using (59) or it can be gained when solving the perturbed Klein Gordon equation:

$$\delta\phi'' + 2\mathcal{H}\delta\phi' + (k^2 + a^2 U_{,\phi\phi})\delta\phi + 4\bar{\phi}'\Phi' - 2a^2 U_{,\phi}\Phi = 3\mathcal{H}^2 \Omega_c C_c [\delta_c - 2\Phi]. \quad (64)$$

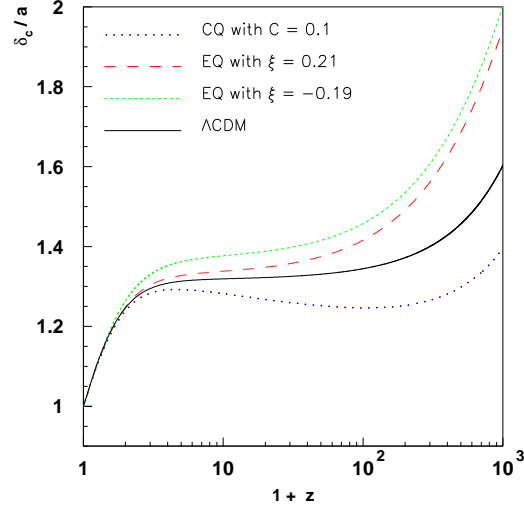


FIG. 3: Growth factor as a function of redshift for coupled dark energy and extended quintessence with a non minimal coupling (for positive and negative couplings). The case of Λ CDM is also shown for reference. The value of ξ corresponds to $\omega_{JBD0} \sim 30$.

In fig.(3) the density perturbation of CDM devided by the scale parameter and normalized to one today is shown; the latter quantity is the cosmological growth factor for CDM. The curves have been obtained by numerical integration and correspond to the case of CQ (dotted) as well as EQ case, for a positive (long dashed) or negative (dashed) coupling, as we will expose in detail in the following. The Λ CDM case is also shown for reference (solid): as it is well known, in the latter scenario the growth factor is almost constant during the Matter Dominated Era (MDE, $\delta_c \propto a$), while it increases going back in time due to the presence of radiation. The cosmological parameters of these scenarios are as defined in the previous section. We postpone further comments to this figure to the end of this section, when we expose the EQ case, too.

B. Extended quintessence

CDM variables follow the usual uncoupled equations, written here in newtonian gauge:

$$\delta'_c = -kv_c - 3\Phi', \quad (65)$$

$$v'_c = -\mathcal{H}v_c + k\Psi. \quad (66)$$

Note that, in contrast with the CQ case, as we will see later, in EQ the two gravitational potentials Φ and Ψ are not equal in module, since the anisotropic stress is in general different from zero. As we show now, however, this is not the major source of difference between the two scenarios.

As shown in [33, 36] the perturbed Einstein equations can still be written formally as

$$\delta G^\mu_\nu = \kappa \delta T^\mu_\nu, \quad (67)$$

where G^μ_ν is the usual Einstein tensor and δT^μ_ν is the total stress energy tensor, summing up all species:

$$\delta T^\mu_\nu = \delta T^\mu_{\nu(f)} + \delta T^\mu_{\nu(\phi)}. \quad (68)$$

In particular the conserved stress energy tensor for ϕ is given by

$$\delta T^\mu_{\nu(\phi)} = \delta T^{\mu mc}_{\nu}[\phi] + \delta T^{\mu nmc}_{\nu}[\phi] + \delta T^{\mu grav}_{\nu}[\phi], \quad (69)$$

where, in the newtonian gauge:

$$\delta T^{\mu mc}_{\nu}[\phi] = \frac{1}{a^2} \left(\Psi \phi'^2 - \phi' \delta \phi' \right) - U_{,\phi} \delta \phi \quad (70)$$

$$\delta T_\nu^{\mu nmc}[\phi] = -\frac{3\mathcal{H}}{a^2} \left[2\Psi - \frac{\Phi'}{\mathcal{H}} \right] F' + \frac{3\mathcal{H}}{a^2} \delta F' + \frac{F_{,\phi} R}{2} \delta\phi + \left(\frac{k^2}{a^2} - \frac{R}{2} \right) \delta F, \quad (71)$$

$$\delta T_\nu^{\mu grav}[\phi] = \frac{3\mathcal{H}^2}{a^2} \delta F + \left(\frac{1}{\kappa} - F \right) \delta G_\nu^\mu. \quad (72)$$

Carrying the last term of expression (72) to the left hand side in the perturbed Einstein equations, we can rewrite eqs(67) as

$$F \delta G_\nu^\mu = \delta T_\nu^{\mu mc}[\phi] + \delta T_\nu^{\mu nmc}[\phi] + \frac{3\mathcal{H}^2}{a^2} \delta F + \delta T_\nu^{\mu (fluid)}. \quad (73)$$

The (0,0) component of (73) reads

$$\begin{aligned} \frac{2k^2}{a^2} \Phi = & \frac{1}{F} \left[\frac{6\mathcal{H}}{a^2} (\mathcal{H}F + F') \Psi - \frac{3}{a^2} (2\mathcal{H}F + F') \Phi' + \right. \\ & + \frac{1}{a^2} \left(\phi' \delta\phi' - \phi'^2 \Psi + a^2 \left(U_{,\phi} - \frac{F_{,\phi} R}{2} \right) \delta\phi + \right. \\ & \left. \left. - 3\mathcal{H} \delta F' - \left(k^2 + 3\mathcal{H}^2 - a^2 \frac{R}{2} \right) \delta F \right) + \rho_f \delta_f \right]. \end{aligned} \quad (74)$$

The (0,j) component of (73) gives the following equations:

$$[2\mathcal{H}F + F'] \Psi - 2F \Phi' = \phi' \delta\phi + \delta F' - \mathcal{H} \delta F + \frac{a^2}{k} \delta T_{j fluid}^0. \quad (75)$$

Combining the two previous equations to get rid of Φ' , we get the Poisson equation in EQ cosmologies, which is

$$\begin{aligned} \frac{2k^2}{a^2} \Phi = & \frac{1}{F} \left[-\frac{1}{a^2} \left(\phi'^2 + \frac{3}{2} \frac{F'^2}{F} \right) \Psi + \right. \\ & + \frac{1}{a^2} \left[3 \left(\mathcal{H} + \frac{F'}{2F} \right) \phi' \delta\phi + \phi' \delta\phi' + \frac{3}{2} \frac{F'}{F} \delta F' + \right. \\ & \left. \left. - \left(k^2 + 6\mathcal{H}^2 - \frac{a^2 R}{2} + \frac{3\mathcal{H}}{2} \frac{F'}{F} \right) \delta F + a^2 \left(U_{,\phi} - \frac{F_{,\phi} R}{2} \right) \delta\phi \right] + \right. \\ & \left. + \rho_f \delta_f + \frac{3}{k} \left(\mathcal{H} + \frac{F'}{2F} \right) h_f v_f \right], \end{aligned} \quad (76)$$

where we have used that in the Newtonian gauge $\delta T_{j fluid}^0 = v_f h_f$. Finally, the Klein Gordon equation which describes the evolution of the EQ scalar field is

$$\begin{aligned} \delta\phi'' + 2\mathcal{H} \delta\phi' + \left[k^2 - \frac{a^2}{2} (F_{,\phi\phi} R - 2U_{,\phi\phi}) \right] \delta\phi = \\ = \phi' (\Psi' - 3\Phi') + \frac{a^2}{2} (F_{,\phi} R - 4U_{,\phi}) \Psi + \frac{a^2}{2} F_{,\phi} \delta R, \end{aligned} \quad (77)$$

where

$$\delta R = -\frac{2}{a^2} \left[(3\mathcal{H}\Psi - 3\Phi')' + 3\mathcal{H} (3\mathcal{H}\Psi - 3\Phi') - (k^2 - 3\mathcal{H}' + 3\mathcal{H}^2) \Psi - 2k^2 \Phi \right]. \quad (78)$$

These equations hold in general for any choice of the function $F(\phi)$. In order to solve the equations numerically, we again set the case of non-minimal coupling, in which (16) is valid.

The results of the numerical integration are shown in fig.(3). As a reference, we normalize the growthfactor to a common value at present time. In the CQ case, the coupling to CDM has the effect of lowering the growthfactor with respect to the Λ CDM case; this means that for a fixed primordial normalization of the perturbations, the CQ structure formation is enhanced with respect to a Λ CDM case, independently on the sign of the coupling constant, as it is known from earlier works [3]. The EQ phenomenology is completely different, as the structure formation may be slower than the Λ CDM case, depending on the sign and magnitude of the coupling constant. This may have interesting consequences for constraining these theories from the observations of cosmological structures in the mildly and full non-linear regimes, as we see in the next section. Note that, consistently with the analysis of the previous section, in the EQ case the departure from the Λ CDM case is again bigger for a negative coupling.

V. NEWTONIAN LIMIT

A coupling between dark energy and matter or gravity could indeed have observable effects on structure formation, due either to the interaction between species which behave differently with respect to gravity or because of a modification of gravity itself. We perform here a derivation of the Newtonian limit for cosmological perturbations in CQ and EQ scenarios [3, 30, 35]. This part of the work represents the interface to numerical simulations of structure formation in these scenarios. Previous works on this subject considered the cases of CQ only [3, 13]. We do expect to be able to track the differences outlined in the previous sections also in the present context.

A. Coupled dark energy

We strictly follow the notation of [3, 13] and we specialize the previous general equations to the case in which $\lambda \equiv \mathcal{H}/k \ll 1$. This choice corresponds to the Newtonian limit, that is to say scales much smaller than the horizon. When the newtonian limit holds, the expressions of both the gravitational potential Φ and its derivative Φ' can be largely simplified and become approximately equal to:

$$\Phi = \frac{1}{2k^2} [\phi'(\delta\phi)' + 3\mathcal{H}\phi'\delta\phi + a^2 U_{,\phi}\delta\phi + 3\Omega_c \mathcal{H}^2 \delta_c] , \quad (79)$$

$$\Phi' = -\frac{1}{2}\phi'\delta\phi - \mathcal{H}\Phi . \quad (80)$$

Substituting Φ' in the perturbed Klein Gordon equation (64), we get the following equation describing the evolution of ϕ :

$$\delta\phi'' + 2\mathcal{H}\delta\phi' + (k^2 + a^2 U_{,\phi\phi} - 2\phi'^2)\delta\phi + \Phi(-4\mathcal{H}\phi' - 2a^2 U_{,\phi} + 6\mathcal{H}^2 \Omega_c C_c) = 3\mathcal{H}^2 \Omega_c C_c \delta_c . \quad (81)$$

In the Newtonian limit $k \gg \mathcal{H}$ and the latter equation can be further simplified: the term containing Φ can be neglected since Φ is order k^{-2} ; within the terms multiplying $\delta\phi$, we neglect the contribution of the potential, which has effect only at very recent times, when the acceleration takes over; furthermore, the remaining piece $k^2 - 2\phi'^2$ can be rewritten as $k^2(1 - 2\lambda^2(d\phi/d\alpha)^2) \sim k^2$ where $\alpha \equiv \log a$. Supposing that also $\delta\phi''$ and $\delta\phi'$ can be neglected (this can be further checked a posteriori), eq.(81) in the newtonian limit reduces to:

$$\delta\phi \sim 3\lambda^2 \Omega_c C_c \delta_c , \quad (82)$$

stating that $\delta\phi$ is of order λ^2 and can therefore be neglected, together with its derivative, in eq.(79) with respect to the last term:

$$\Phi \sim \frac{3}{2}\lambda^2 \Omega_c \delta_c . \quad (83)$$

By looking at the right hand side of eq.(50) we can formally define the quantity:

$$\Phi_c \equiv \Phi + C_c \delta_\phi , \quad (84)$$

which reads

$$\Phi_c = \frac{3}{2}\lambda^2 \Omega_c \delta_c (1 + 2C_c^2) , \quad (85)$$

where we have used the approximated expression (82) for $\delta\phi$ obtained by solving the KG equation in the newtonian limit. The latter equation reads in real space as

$$\nabla^2 \Phi_c = -\frac{a^2}{2} \rho_c \delta_c (1 + 2C_c^2) , \quad (86)$$

and if we explicit the units contained in ρ^2 we then have that Φ_c behaves like a new gravitational potential satisfying:

$$\nabla^2 \Phi_c = -4\pi G_* a^2 \rho_c \delta_c , \quad (87)$$

² Recall that our energy densities are equal to $\hat{\rho} = 8\pi G\rho$ and we have redefined $\hat{\rho} = \rho$ for simplicity.

where we have defined G_* as

$$G_* \equiv G(1 + 2C_c^2) . \quad (88)$$

In this way, we formally recover the usual Poisson equation, where the effect of the coupling is partially included in the redefinition of the gravitational potential Φ_c and partially in a varying gravitational constant G_* .

The equations for the CDM component (55 and 50) can also be analogously simplified and read, in the Newtonian limit, as:

$$\delta_c' = -kv_c , \quad (89)$$

$$v_c' + (\mathcal{H} - C_c\bar{\phi}')v_c = -k\Phi_c . \quad (90)$$

Deriving the first equation and combining it with the second, it is straightforward to obtain

$$\delta_c'' + (\mathcal{H} - C_c\phi')\delta_c' - \frac{3}{2}\mathcal{H}^2\Omega_c\delta_c(1 + 2C_c^2) = 0 . \quad (91)$$

Furthermore, equation (90) allows us to clarify how the Euler equation, relevant for the interaction of dark matter particles within structure formation, changes due to the presence of a coupling between dark matter and dark energy (Einstein frame). Taking into account expression (84) and rewriting (90) in real space coordinates, the equation finally reads as a modified Euler equation:

$$\nabla v_c' + (\mathcal{H} - C_c\phi')\nabla v_c + \frac{3}{2}\mathcal{H}^2\Omega_c\delta_c(1 + 2C_c^2) = 0 . \quad (92)$$

Supposing now that CDM is concentrated in one particle of mass m_c at a distance r from a particle of mass M_c at the origin we can rewrite the cold dark matter density contribution as

$$\Omega_c\delta_c = \frac{8\pi GM_c e^{-C_c(\phi-\phi_0)}\delta(0)}{3\mathcal{H}^2 a} , \quad (93)$$

where we have used the fact that a non-relativistic particle at position r has a density given by $m_c n \delta(r)$ (where $\delta(r)$ stands for the Dirac distribution) with number density $n = e^{-C_c(\phi-\phi_0)}$ given by equation (34) and we have assumed that the density of the M_c mass particle is much larger than ρ_c . The Euler equation in cosmic time $dt = a d\tau$ then reads:

$$\nabla \dot{v}_c = -\tilde{H}\nabla v_c - \frac{4\pi\tilde{G}M_c\delta(0)}{a^2} , \quad (94)$$

where we included the mass scaling in a new redefinition of the gravitational constant

$$\tilde{G} = G_* e^{-C_c(\phi-\phi_0)} = G(1 + 2C_c^2)e^{-C_c(\phi-\phi_0)} , \quad (95)$$

and the further effect of the coupling on the friction term is a redefinition of the Hubble constant:

$$\tilde{H} = H \left(1 - C_c \frac{\dot{\phi}}{H} \right) . \quad (96)$$

Though the choice of including the exponential correction in a redefinition of the gravitational constant might be convenient for numerical purposes, we can as well rewrite eq.(94) as:

$$\nabla \dot{v}_c = -\tilde{H}\nabla v_c - \frac{4\pi G_* \tilde{M}_c \delta(0)}{a^2} , \quad (97)$$

in which gravity is still governed by a gravitational (though modified) constant and the time dependent extra contribution is included in the mass $\tilde{M}_c \equiv M_c e^{-C_c(\phi-\phi_0)}$ of dark matter: when dark matter particles interact with each other, each dark matter particle sees the other one as possessing a ‘modified’ mass which takes into account the non-standard scenario due to the coupling.

Along the attractor (36), the full correction to the gravitational constant and to the expansion assume the following

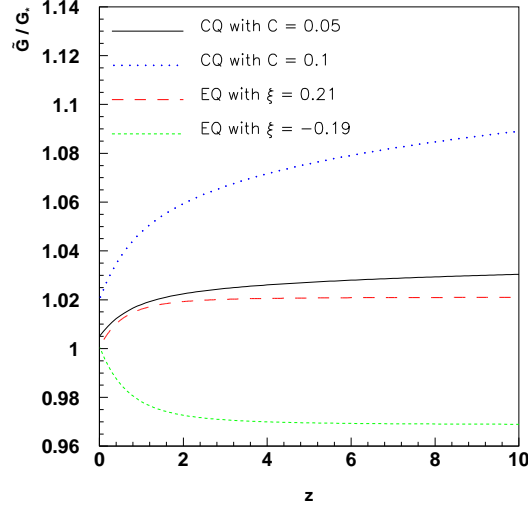


FIG. 4: Gravitational correction as a function of redshift for CQ and EQ models. The value of ξ corresponds to $\omega_{JBD_0} \sim 30$.

expressions:

$$\tilde{G} = G(1 + 2C_c^2)a^{-2C_c^2}, \quad (98)$$

$$\tilde{H} = H(1 - 2C_c^2), \quad (99)$$

$$\tilde{M} = M_c a^{-2C_c^2}. \quad (100)$$

This clearly shows how the corrections behave regardless of the sign of the coupling constant. At the end of this Section, we'll focus on this aspect again, comparing the corrections to the strength of gravity in CQ and EQ.

B. Extended quintessence

In the limit of small scales, in which $k \gg \mathcal{H}$ we can proceed in simplifying the equations for EQ in a way which is analogous to the one adopted in CQ. Neglecting time derivatives (negligible in the newtonian limit with respect to spatial derivatives) and the potential, the Poisson equation (76) reduces to:

$$\frac{2k^2}{a^2}F\Phi = -\frac{k^2}{a^2}F_{,\phi}\delta\phi + \rho_f\delta_f, \quad (101)$$

which reads, in real space, as:

$$F\nabla^2\Phi = -4\pi Ga^2\rho_f\delta_f - \frac{F_{,\phi}}{2}\nabla^2\delta\phi. \quad (102)$$

From the traceless part of the perturbed Einstein equations (component (i,j)) we get the gravitational potential relation

$$(\Psi + \Phi) = -\frac{\delta F}{F}, \quad (103)$$

so that in general the right hand side, representing the anisotropic stress, is different from zero in scalar tensor theories. Equation (102) can then be equivalently written, in terms of Ψ , as

$$F\nabla^2\Psi = 4\pi Ga^2\rho_f\delta_f - \frac{F_{,\phi}}{2}\nabla^2\delta\phi. \quad (104)$$

In the approximation we are considering, the perturbed Klein Gordon equation given by (77) only receives contribution from

$$\delta\phi = F_{,\phi} [\Psi + 2\Phi] . \quad (105)$$

Substituting (103), we obtain:

$$\delta\phi = \frac{FF_{,\phi}}{F + F_{,\phi}^2} \Phi = -\frac{FF_{,\phi}}{F + 2F_{,\phi}^2} \Psi . \quad (106)$$

Further substituting in (101), the Poisson equation can be rewritten in the usual way

$$\frac{2k^2}{a^2} \Phi_{\mathbf{E}} = \frac{1}{F} \rho_f \delta_f , \quad (107)$$

or, in real space, as:

$$\nabla^2 \Phi_{\mathbf{E}} = -\frac{4\pi G}{F} a^2 \rho_f \delta_f , \quad (108)$$

if we redefine the gravitational potential as

$$\Phi_{\mathbf{E}} = \left(1 + \frac{1}{2} \frac{F_{,\phi}^2}{F + F_{,\phi}^2} \right) \Phi . \quad (109)$$

Note that we could have equivalently substitute $\delta\phi(\Psi)$ in (104) thus obtaining

$$\nabla^2 \Psi_{\mathbf{E}} = \frac{4\pi G}{F} a^2 \rho_f \delta_f , \quad (110)$$

if we redefine the gravitational potential as:

$$\Psi_{\mathbf{E}} = \left(1 - \frac{1}{2} \frac{F_{,\phi}^2}{F + 2F_{,\phi}^2} \right) \Psi . \quad (111)$$

The Euler equation (66) is then modified according to:

$$v'_c + \mathcal{H}v_c + k \left(1 + \frac{F_{,\phi}^2}{F + F_{,\phi}^2} \right) \Phi = 0 . \quad (112)$$

Substituting the expression for $\Phi(\Phi_{\mathbf{E}})$ given by (109) and making use of the Poisson equation (107) we get

$$v'_c + \mathcal{H}v_c + \frac{a^2}{2k} \rho_f \delta_f \frac{2(F + 2F_{,\phi}^2)}{F(2F + 3F_{,\phi}^2)} = 0 , \quad (113)$$

which in real space reads as:

$$\nabla v'_c + \mathcal{H}\nabla v_c + \frac{3}{2} \mathcal{H}^2 \Omega_f \delta_f \frac{2(F + 2F_{,\phi}^2)}{F(2F + 3F_{,\phi}^2)} = 0 , \quad (114)$$

where we have defined $\Omega_f \equiv \rho_f / \rho_{crit}$ and $\rho_{crit} \equiv 3\mathcal{H}^2 F / a^2$. For comparison with the CQ case investigated in the previous paragraph, suppose now that the only other fluid present is CDM, concentrated in one particle of mass m_c at a distance r from a particle of mass M_c at the origin, such that

$$\Omega_c \delta_c = \frac{1}{F} \frac{M_c \delta(0)}{3\mathcal{H}^2 a} , \quad (115)$$

due to the fact that in this framework cold dark matter is uncoupled from dark energy and approximately scales in the usual way $\rho_c \propto a^{-3}$. Substituting the latter expression in (114) we get

$$\nabla \dot{v}_c + H\nabla v_c + \frac{4\pi \tilde{G} M_c \delta(0)}{a^2} = 0 , \quad (116)$$

in terms of the cosmic time, where we have included the extra contribution in the redefinition of the gravitational constant, which is now varying in time:

$$\tilde{G} = \frac{2(F + 2F_{,\phi}^2)}{(2F + 3F_{,\phi}^2)} \frac{1}{8\pi F} . \quad (117)$$

The latter formalism is general for any choice of $F(\phi)$. For illustration we specify our final results to the non minimal coupling choice given by (16). In this case

$$\tilde{G} = \frac{\left[\frac{1}{8\pi G_*} + (1 + 8\xi)\xi\phi^2 \right]}{\left[\frac{1}{8\pi G_*} + (1 + 6\xi)\xi\phi^2 \right]} \frac{1}{\left[\frac{1}{G_*} + 8\pi\xi\phi^2 \right]} . \quad (118)$$

For small values of the coupling, that is to say $\xi \ll 1$ the latter expression becomes:

$$\frac{\tilde{G}}{G_*} \sim 1 - 8\pi G_* \xi \phi^2 \quad (119)$$

which manifestly depends on the sign of the coupling ξ . In fig.(4) we compare the behavior of the correction to the gravitational constant in CQ and EQ theories, for different values and sign of the coupling constants. As we stressed in this Section, in the CQ case the correction is independent on the sign of the coupling constant, just as expected since the theory corresponds to the case of induced gravity, which, as discussed in the first section, forces the sign dependence in order to maintain gravity as an attractive force; note also how also at present the value of the gravitational coupling is larger than the corresponding Λ CDM case, as it is evident from (98). On the other hand, in the EQ case, the sign has the effect of increasing or decreasing the gravitational strength.

C. Recipe for Nbody users

We conclude this section by summarizing the quantities needed as an input for N-body simulations when considering CQ models or EQ models as discussed in this paper. An N-body code which wants to take into account the modified gravitational strength between different species, as well as the modified perturbation growth rate, needs to read tables with the modification listed in table I for both sets of cosmologies. The expansion history, represented by the Hubble expansion rate, needs also to be modified, although it is not listed in the table.

Three typologies of corrections are considered: the gravitational interaction between dark matter (DM) particles, the friction term due to a modified expansion and the mass of the interacting species.

When dealing with CQ models, all three corrections are active: the gravitational interaction between DM-DM particles changes, the friction term provides an extra contribution due to the coupling and the mass of the DM particles also changes with time and depending of the coupling. In the case in which we also consider baryons (B), non coupled to the scalar field ϕ or to CDM, the gravitational interaction between DM-B or B-B is the usual Newtonian one, since non coupled baryons satisfy the standard Euler equation. Furthermore, no correction to the baryon mass is required. On the contrary, when EQ cosmologies are considered, only the gravitational correction needs to be applied. The effect of the coupling is all included in an effective gravitational ‘constant’, which changes with time according to eq.(118) for all the species considered. Both the friction term and the mass term remain unaltered.

VI. CONCLUSIONS

In this paper we performed a comparative analysis of the theoretical framework and phenomenology of cosmological perturbations in two of the main scenarios where the dark energy is explicitly coupled with dark matter (Coupled Quintessence, CQ) or gravity (Extended Quintessence, EQ). Weyl scaling the two theories, we fixed the expressions of the two scenarios in the Jordan and Einstein frames. This showed how they lead to markedly different phenomenologies, even if in the Jordan frame, they differ only by a constant term into the function multiplying the Ricci scalar: while the CQ case is bounded to produce effective corrections to gravity which increase its strength, the EQ case is able to provide corrections in both senses. This has important consequences which we highlight in the rest of the paper.

We show that indeed linear perturbation growth may actually accelerate or slow down in EQ cases with respect to a case in which the dark energy is uncoupled, while in CQ scenarios perturbations can be enhanced only. In order to set

Correction type	CQ	EQ
Gravity ($\Delta G/G_*$) DM-DM	$1 + 2C_c^2$	\tilde{G}/G_*
B-DM	1	\tilde{G}/G_*
B-B	1	\tilde{G}/G_*
Friction ($\Delta\mathcal{H}/\mathcal{H}$)	$1 - \frac{C_c\phi'}{\mathcal{H}}$	1
Mass ($\Delta m/m$) DM	$e^{-C_c(\phi-\phi_0)}$	1
B	1	1

TABLE I: Summary of corrections required to run N-body simulations in CQ and EQ scenarios.

up a suitable framework for the realization of N-body simulations in these cosmologies, we derive the Newtonian limit for both, again highlighting the different behavior that the gravitational corrections possess, never neglecting the role of Quintessence fluctuations. Finally, we list the quantities which need to be modified in N-body codes for taking into account the corrections to the expansion, strength of gravity as perceived by different species, as well as the modified perturbation growth rate, in body scenarios, when performing simulations of non-linear structure formation.

Our work opens up the possibility of measuring the effect of non-minimal dark energy models in numerical simulations of structure formation, thus being able to constrain them through observations. N-body simulations nowadays reach the extension of thousands of Mpc, where indeed the interplay between large and small scale behavior of gravity really matters. Possible deviations from the ordinary behavior of gravity in General Relativity, which might be consistent with non-minimal dark energy models have to be detected coherently on all scales in order to provide convincing evidence. This capability can be built only by understanding properties and differences between the ways in which such deviations could manifest, developing capabilities for simulating and detecting them in large scale numerical simulations of structure formation and finally constraining them through observations. This work completes the first step in this path.

Acknowledgments

We are grateful to F. Perrotta and J. Macher for helpful collaboration. We thank E.Linder, K.Dolag, L.Moscardini, M.Bartelmann, S.Matarrese, M.Meneghetti, C.Wetterich, G.Robbers for useful discussions. This work was supported by Progetto D4 Regione Friuli Venezia Giulia. VP is supported by the Alexander von Humboldt Foundation.

References

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- [1] L. Amendola, Phys. Rev. D **62** (2000) 043511 [arXiv:astro-ph/9908023].
 - [2] S. Matarrese, M. Pietroni and C. Schmid, JCAP **0308** (2003) 005 [arXiv:astro-ph/0305224].
 - [3] L. Amendola, Phys. Rev. D **69** (2004) 103524 [arXiv:astro-ph/0311175].
 - [4] S. Lee, G. C. Liu and K. W. Ng, Phys. Rev. D **73** (2006) 083516 [arXiv:astro-ph/0601333].
 - [5] B. Wang, J. Zang, C. Y. Lin, E. Abdalla and S. Micheletti, Nucl. Phys. B **778** (2007) 69 [arXiv:astro-ph/0607126].
 - [6] R. Mainini and S. Bonometto, arXiv:0709.0174 [astro-ph].
 - [7] L. Amendola, M. Gasperini and F. Piazza, arXiv:astro-ph/0407573.
 - [8] L. Amendola, G. Camargo Campos and R. Rosenfeld, arXiv:astro-ph/0610806.
 - [9] Z. K. Guo, N. Ohta and S. Tsujikawa, Phys. Rev. D **76** (2007) 023508 [arXiv:astro-ph/0702015].
 - [10] R. Mainini and S. Bonometto, JCAP **0706** (2007) 020 [arXiv:astro-ph/0703303].
 - [11] E. Abdalla, L. R. W. Abramo, L. Sodre and B. Wang, arXiv:0710.1198 [astro-ph].
 - [12] O. Bertolami, F. G. Pedro and M. L. Delliou, arXiv:0705.3118 [astro-ph].
 - [13] A. V. Maccio, C. Quercellini, R. Mainini, L. Amendola and S. A. Bonometto, Phys. Rev. D **69** (2004) 123516 [arXiv:astro-ph/0309671].
 - [14] A. W. Brookfield, C. van de Bruck and L. M. H. Hall, arXiv:0709.2297 [astro-ph].
 - [15] G. Mangano, G. Miele and V. Pettorino, Mod. Phys. Lett. A **18** (2003) 831 [arXiv:astro-ph/0212518].

- [16] T. Koivisto, Phys. Rev. D **72** (2005) 043516 [arXiv:astro-ph/0504571].
- [17] M. Quartin, M. O. Calvao, S. E. Joras, R. R. Reis and I. Waga, arXiv:0802.0546 [astro-ph].
- [18] G. Olivares, F. Atrio-Brandela and D. Pavon, arXiv:0801.4517 [astro-ph].
- [19] C. G. Boehmer, G. Caldera-Cabral, R. Lazkoz and R. Maartens, arXiv:0801.1565 [gr-qc].
- [20] R. Fardon, A. E. Nelson and N. Weiner, JCAP **0410** (2004) 005 [arXiv:astro-ph/0309800].
- [21] A. W. Brookfield, C. van de Bruck, D. F. Mota and D. Tocchini-Valentini, Phys. Rev. D **73** (2006) 083515 [arXiv:astro-ph/0512367].
- [22] L. Amendola, M. Baldi and C. Wetterich, arXiv:0706.3064 [astro-ph].
- [23] C. Wetterich, Phys. Lett. B **655** (2007) 201 [arXiv:0706.4427 [hep-ph]].
- [24] D. F. Mota, V. Pettorino, G. Robbers and C. Wetterich, Phys. Lett. B **663** (2008) 160 arXiv:0802.1515 [astro-ph].
- [25] R. Bean, E. E. Flanagan and M. Trodden, arXiv:0709.1128 [astro-ph].
- [26] N. Afshordi, M. Zaldarriaga and K. Kohri, Phys. Rev. D **72** (2005) 065024 [arXiv:astro-ph/0506663].
- [27] O. E. Bjaelde, A. W. Brookfield, C. van de Bruck, S. Hannestad, D. F. Mota, L. Schrempf and D. Tocchini-Valentini, JCAP **0801** (2008) 026 arXiv:0705.2018 [astro-ph].
- [28] J. C. Hwang, Class. Quant. Grav. **7**, 1613 (1990).
- [29] J. C. Hwang, Phys. Rev. D **42** (1990) 2601.
- [30] C. Schmid, J. P. Uzan and A. Riazuelo, Phys. Rev. D **71** (2005) 083512 [arXiv:astro-ph/0412120].
- [31] A. Riazuelo and J. P. Uzan, Phys. Rev. D **66** (2002) 023525 [arXiv:astro-ph/0107386].
- [32] J. P. Uzan, Phys. Rev. D **59** (1999) 123510 [arXiv:gr-qc/9903004].
- [33] V. Faraoni, Phys. Rev. D **62** (2000) 023504 [arXiv:gr-qc/0002091].
- [34] F. Perrotta, C. Baccigalupi and S. Matarrese, Phys. Rev. D **61**, 023507 (2000)
- [35] B. Boisseau, G. Esposito-Farese, D. Polarski and A. A. Starobinsky, Phys. Rev. Lett. **85** (2000) 2236 [arXiv:gr-qc/0001066].
- [36] F. Perrotta and C. Baccigalupi, Phys. Rev. D **65** (2002) 123505 [arXiv:astro-ph/0201335].
- [37] S. Matarrese, C. Baccigalupi and F. Perrotta, Phys. Rev. D **70** (2004) 061301 [arXiv:astro-ph/0403480].
- [38] V. Pettorino, C. Baccigalupi and F. Perrotta, JCAP **0512** (2005) 003 [arXiv:astro-ph/0508586].
- [39] V. Pettorino, C. Baccigalupi and G. Mangano, JCAP **0501** (2005) 014 [arXiv:astro-ph/0412334].
- [40] F. Perrotta, S. Matarrese, M. Pietroni and C. Schmid, Phys. Rev. D **69** (2004) 084004 [arXiv:astro-ph/0310359].
- [41] Wands D. 1994, Class. Quant. Grav. **11**, 269 gr-qc/9307034
- [42] Esposito-Farese G. Polarski D. 2001, Phys. Rev. D **63**, 063504, gr-qc/0009034
- [43] Wetterich C. 1988 Nucl. Phys. B **302**, 645
- [44] K. i. Maeda, Phys. Rev. D **39**, 3159 (1989).
- [45] R. Catena, M. Pietroni and L. Scarabello, Phys. Rev. D **76** (2007) 084039 [arXiv:astro-ph/0604492].
- [46] Doran M. Jaekel J. 2002, Phys. Rev. D **66** 043519, astro-ph/0203018
- [47] A. Zee, Phys. Rev. Lett. **42** (1979) 417.
- [48] D. J. Holden and D. Wands, Phys. Rev. D **61** (2000) 043506 [arXiv:gr-qc/9908026].
- [49] Kodama H., Sasaki M. 1984, Prog. Theor. Phys. Suppl. **78**, 1
- [50] B. Ratra, P. J. E. Peebles 1988, Astrophys. J. **325**, L17
- [51] Brax P., Martin J. 2000, Phys. Rev. D **61** 103502 astro-ph/9912046
- [52] Zlatev I., Wang L., Steinhardt P.J. 1999, Phys. Rev. Lett. **82**, 896 astro-ph/9807002
- [53] D. N. Spergel *et al.* 2003, Astrophys. J. Supp. **148**, 175 arXiv:astro-ph/0302209
- [54] M. Tegmark *et al.* 2004, Phys. Rev. D **69** 103501 arXiv:astro-ph/0310723
- [55] A. G. Riess *et al.* 2004, arXiv:astro-ph/0402512
- [56] D. N. Spergel *et al.*, arXiv:astro-ph/0603449.
- [57] J. J. Halliwell, Phys. Lett. B **185** (1987) 341.
- [58] C. Wetterich, Astron. Astrophys. **301** (1995) 321 arXiv:hep-th/9408025.
- [59] V. Faraoni, arXiv:gr-qc/0703044.
- [60] Bertotti B., Iess L., Tortora P. 2003, Nature **425**, 374
- [61] V. Acquaviva, C. Baccigalupi, S. M. Leach, A. R. Liddle and F. Perrotta, Phys. Rev. D **71** (2005) 104025 [arXiv:astro-ph/0412052].
- [62] T. Clifton, D. F. Mota and J. D. Barrow, Mon. Not. Roy. Astron. Soc. **358** (2005) 601 [arXiv:gr-qc/0406001].
- [63] C. Baccigalupi, S. Matarrese and F. Perrotta, Phys. Rev. D **62** (2000) 123510 [arXiv:astro-ph/0005543].